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A vehicle routing problem with a predefined customer sequence, stochastic demands and penalties for unsatisfied demands

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Abstract. We consider a stochastic vehicle routing problem in which the customers are served according to a predefined sequence and the demands of the customers are discrete random variables. It is assumed that a penalty cost is imposed if a customer's demand is not satisfied or if it is satisfied partially. The objective is the determination of the policy that serves the customers with the minimum total expected cost. A suitable dynamic programming algorithm is developed for the determination of the optimal policy. It is proved that the optimal policy has a specific threshold-type structure.

Keywords: routing problem; partial service; dynamic programming

Introduction

The Vehicle Routing Problem (VRP) is a combinatorial optimization problem with significant applications in the fields of transportation, distribution and logistics. It consists of designing the optimal delivery routes of a fleet of vehicles that originate from one or several depots and deliver goods to N geographically scattered customers comprising the nodes of a predefined network. Four interesting variants of the VRP that have been studied extensively in the literature are: (i) the VRP with time windows in which the delivering locations have time windows within which the deliveries must be made, (ii) the capacitated VRP (with or without time windows) in which the vehicles have limited carrying capacity, (iii) the VRP with backhauls in which the customers are divided into linehaul customers, each requiring a given quantity of product to be delivered, and backhaul customers, where a given quantity of products must be picked up and (iv) the VRP with pickup and delivery in which each customer is associated with two quantities representing the demands of products to be delivered and picked up. Suitable exact algorithms (e.g. branch-and-bound, branch-and-cut, branch-and-cut-and-price) and heuristics or metaheuristics (tabu search, simulated annealing, genetic algorithms, colony optimization) have been developed. The exact algorithms find the global minimum for the cost function. The heuristics and metaheuristics search for "good solutions" and in many cases the cost of their final routing strategy is equal to the global minimum. Surveys of models and solutions that are related to various versions of the VRP are presented in Toth and Vigo (2002), Simchi-Levi, Chen, and Bramel (2005), and Liong et al (2008).

In the present paper we study a simple capacitated VRP in which a single vehicle with finite capacity starts its route from a depot and delivers a product to N customers according to a predefined sequence $1 \to \cdots \to N.$ The demand of the customer $j \in \{1,\ldots,N\}$ for the product is a discrete random variable. The actual demand of each customer is revealed only upon the vehicle's visit to the customer. We assume that it is permissible to satisfy partially or not to satisfy the demand of the customer if his/her demand exceeds the load of the vehicle. In this case a penalty cost is incurred. The vehicle after serving fully or partially the customers returns to the depot. Our objective is to find the routing strategy that minimizes the expected total cost during a visit cycle. This cost includes travel costs between consecutive customers, travel costs between customers and the depot and penalty costs due to unsatisfied demands. We present a dynamic programming algorithm that computes the optimal routing strategy. We choose as decision epochs of the problem the epochs at which the vehicle visits for the first time each customer and has satisfied as much of the customer's demand as possible. We prove that the optimal policy has a specific threshold-type structure that is characterized by three critical numbers. Note that stochastic vehicle routing problems with a predefined customer sequence have been considered by Yang et al. (2000) and Pandelis et al. (2012, 2013).

The routing problem and the optimal routing strategy

Consider a set of nodes $V = \{0,1,\ldots,N\}$ with node 0 denoting the depot and the nodes $1,\ldots,N$ corresponding to customers. The customers are serviced in the order $1, 2,\ldots, N$ by a vehicle. Let ξ_j, $j \in \{1,\ldots,N\}$, be the number of items of a particular product that customer j demands. It is assumed that ξ_j is a discrete random variable with possible values $0, 1,\ldots, Q$, where Q is the capacity of the vehicle. The probability distribution of ξ_j, $j \in \{1,\ldots,N\}$ is assumed to be known. The actual demand of each customer is revealed only upon the vehicle's arrival at the customer's site. We denote by $c_{j,j+1}$, $j = 1,\ldots, N-1$, the travel cost between customer *j* and customer $j+1$ and by c_{j0}, c_{0j}, $j = 1,\ldots, N$, the travel cost between customer *j* and the depot and the travel cost between the depot and customer *j*, respectively. These costs can be considered as the costs of the required driver's labor and of the gasoline that the vehicle needs to cover the distances between successive customers or the distances between customers and the depot. It is reasonable to assume that these costs are symmetric and satisfy the triangle inequality, i.e. $c_{j0} = c_{0j}$, $j = 1,\ldots, N$, and $c_{j,j+1} \le c_{j0} + c_{0,j+1}$, $j = 1,\ldots, N-1$. The road network is depicted in Figure 1.

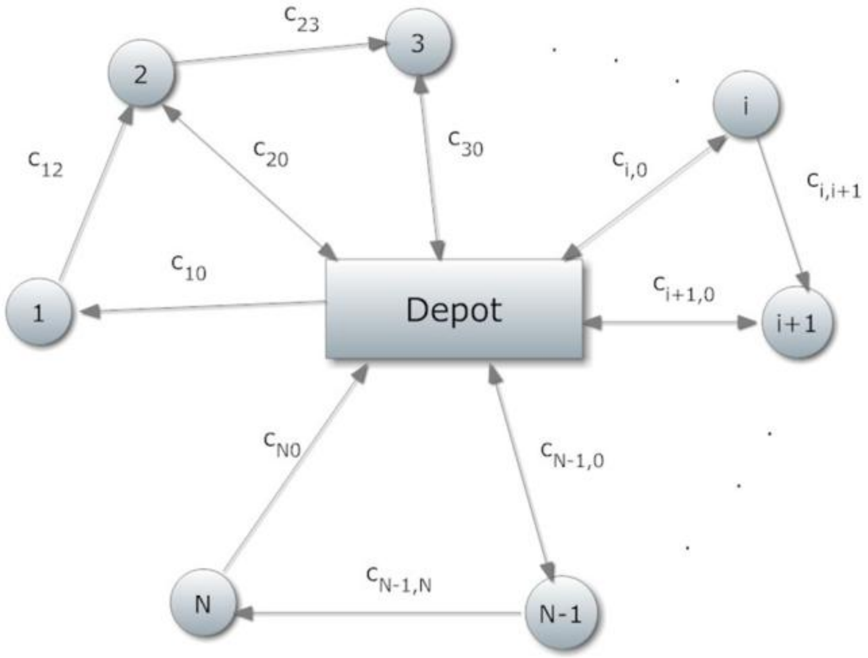

Fig. 1. The road network for the problem

Assume that the vehicle starts its route from the depot loaded with *Q* items of the product. When the vehicle visits a customer $j \in \{1,\ldots, N\}$ for the first time it satisfies as much demand as possible. Let $z \in \{-Q,\ldots, Q\}$ be the number of items of the product carried by the vehicle after the first visit at customer's *j* site. Negative values of *z* denote the unsatisfied demand. This is not possible for $j = 1$ since the vehicle starts its route with *Q* items. For $j \in \{1,\ldots, N-1\}$ and $z \in \{0,\ldots, Q-1\}$, the vehicle either (i) proceeds directly to next customer $j+1$ (action 1) or goes to the depot, restocks with load *Q* and then visits next customer $j+1$ (action 2). When $z = Q$ the only reasonable action is to proceed directly to next customer (action 1). For $j \in \{2,\ldots, N-1\}$ and $z \in \{-Q,\ldots,-1\}$ it has four choices: (i) to proceed directly to next customer $j+1$ (action 1), (ii) to go to the depot to restock with load Q and then visit next customer $j+1$ (action 2), (iii) to go to the depot to restock with load Q, return to customer j to deliver $\theta \in \{1,\ldots,-z\}$ items of the product, and then proceed to next customer $j+1$ with $Q-\theta$ items of the product (action 3) and (iv) to go to the depot to restock the owed quantity $-z$, return to customer j, deliver the owed quantity, make a second trip to the depot to restock with load Q, and then proceed to next customer $j+1$ (action 4). It is assumed that there is no extra demand when the vehicle returns to customer j, i.e. ξ_j remains unaltered. Note that action 1 is included as a possible action in this case since it may favorable if $c_{j,j+1}$ is relatively small and ξ_{j+1} is zero. When action 1 or action 2 or action 3 (with $\theta < -z$) is selected, a penalty cost is incurred that is associated with unsatisfied demand. It is assumed that this penalty cost is equal to $\pi_j > 0$ per item. Therefore, if action 1 or action 2 is selected the penalty cost is equal to $-z\pi_j$, while, if action 3 is selected the penalty cost is equal to $(-z-\theta)\pi_j$. If $j = N$ and $z \in \{0,\ldots,Q\}$ the only possible action is to go to the depot to complete the visit cycle. If $z \in \{-Q,\ldots,-1\}$ there are two options: (i) to go to the depot to complete the visit cycle and (ii) to go to the depot to restock the owed quantity $-z$, return to customer N, deliver the owed quantity and then go to the depot to complete the visit cycle. If option (i) is selected a penalty cost that is equal to $-z\pi_N$ is incurred. Our objective is to determine a vehicle routing and replenishment strategy that minimizes the expected total cost during a visit cycle.

Let $f_j(z), z = -Q,\ldots,Q$, denote the minimum expected future cost until the completion of a visit cycle when the number of items of the product carried by the vehicle after visiting customer $j \in \{1,\ldots,N\}$ for the first time is equal to z. Thus, an optimal routing strategy can be determined by the following dynamic programming equations (see e.g. Eq. (6.5) in Bather's (2000) book):

For $j = 1,\ldots,N-1$ we have

$$f_j(Q) = c_{j,j+1} + Ef_{j+1}(Q-\xi_{j+1}),$$
$$f_j(z) = \min\{c_{j,j+1} + Ef_{j+1}(z-\xi_{j+1}),\ c_{j0} + c_{0,j+1} + Ef_{j+1}(Q-\xi_{j+1})\},\ z = 0,\ldots,Q-1. \quad (1)$$

For $j = 2,\ldots,N-1$ we have

$$f_j(z) = \min\{A_j(z), B_j(z), C_j(z), D_j\},\ \ z = -1,\ldots,-Q, \quad (2)$$

where,

$$A_j(z) = c_{j.j+1} - z\pi_j + Ef_{j+1}(-\xi_{j+1}),$$
$$B_j(z) = c_{j0} + c_{0,j+1} - z\pi_j + Ef_{j+1}(Q-\xi_{j+1}),$$
$$C_j(z) = \min_{0<\theta\le -z}[2c_{j0} + c_{j,j+1} - (z+\theta)\pi_j + Ef_{j+1}(Q-\theta-\xi_{j+1})],$$
$$D_j = 3c_{j0} + c_{0,j+1} + Ef_{j+1}(Q-\xi_{j+1}).$$

In the boundary we have

$$f_N(z) = c_{N0} + \min\{2c_{N0}, -z\pi_N\}1(z < 0), \ z \in \{-Q, \ldots, Q\}.$$

The minimum total expected cost during a visit cycle is equal to

$$f_0 = c_{01} + Ef_1(Q - \xi_1).$$

In the above equations the expected values are taken with respect to the random variables ξ_j, $j = 1, \ldots, N$. The first term in the curly brackets in (1) corresponds to action 1 while the second term corresponds to action 2. The first term in the curly brackets in (2) corresponds to action 1, the second term corresponds to action 2, the third term corresponds to action 3 and the fourth term corresponds to action 4. It is possible to prove by induction on *j* the following lemma.

Lemma 1. $f_j(z)$, $j = 1, \ldots, N$, is non-increasing in $z \in \{-Q, \ldots, Q\}$.

A consequence of Lemma 1 is the following theorem that gives a characterization of the optimal routing strategy.

Theorem 1.

i) For each customer $j \in \{1, \ldots, N-1\}$ there exists a critical integer $s_1(j) \geq 0$ such that it is optimal to select action 1 if $z \geq s_1(j)$, while if $0 \leq z < s_1(j)$ it is optimal to select action 2.

ii) For each customer $j \in \{2, \ldots, N-1\}$ there exist two critical integers $s_2(j), s_3(j)$ (with $s_3(j) \leq s_2(j) < 0$) such that if $s_2(j) < z < 0$ the optimal action is one of the actions 1 and 2, if $s_3(j) \leq z \leq s_2(j)$ the optimal action is action 3 and if $z < s_3(j)$ the optimal action is action 4.

Numerical examples

(a) Suppose that $N = 5$ and $Q = 10$. The travel costs between customers j and $j+1$, $j = 1, \ldots, 4$, are given by: $c_{12} = 5$, $c_{23} = 7$, $c_{34} = 6$ and $c_{45} = 5$. The travel costs between customers j, $j = 1, \ldots, 5$, and the depot are given by: $c_{10} = 6$, $c_{20} = 10$, $c_{30} = 8$, $c_{40} = 7$ and $c_{50} = 9$. Note that these costs satisfy the triangle inequality. The penalty costs for customers $j \in \{2,3,4,5\}$ are given by $(\pi_2, \pi_3, \pi_4, \pi_5) = (2, 1.7, 1.9, 2)$. We further assume that for each customer $j = 1, \ldots, 5$, the demand ξ_j of the customer *j* for the product has the following probability mass function:

$$\Pr(\xi_j = x) = \left(\sum_{i=0}^{Q} e^{-\lambda} \frac{\lambda^i}{i!}\right)^{-1} e^{-\lambda} \frac{\lambda^x}{x!}, \ x = 0, \ldots, Q.$$

For $\lambda = 2$, we implemented the dynamic programming algorithm and we found that the optimal routing strategy is characterized by the critical numbers $s_i(j), i = 1, 2, 3$, presented in Table 1, for each customer $j \in \{1, 2, 3, 4\}$.

Table 1. The critical numbers for each customer

j	$s_1(j)$	$s_2(j)$	$s_3(j)$
1	2	-	-
2	1	-10	-9
3	0	-6	-10
4	0	-6	-10

In Table 2 below, we present the actions selected by the optimal policy for each customer j, $j=1,\ldots,4$. Each row of the table corresponds to each customer j, $j=1,\ldots,4$, and each column of the table corresponds to the number $z \in \{-10,\ldots,0,\ldots,10\}$ of items of the product.

Table 2. The actions selected by the optimal policy

j	*-10*	*-9*	*-8*	*-7*	*-6*	*-5*	*-4*	*-3*	*-2*	*-1*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
1	-	-	-	-	-	-	-	-	-	-	2	2	1	1	1	1	1	1	1	1	1
2	4	2	2	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1
3	3	3	3	3	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4	3	3	3	3	3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

The value of the minimum total expected cost during a visit cycle is equal to $f_0 = c_{01} + Ef_1(Q-\xi_1) = 40.441$. In Table 3 we present the optimal value of θ for customer $j \in \{3, 4\}$ when the optimal action is action 3. The symbol "-" indicates that action 3 is not optimal.

Table 3. The value of optimal θ for 3rd and 4th customer

j\z	*-10*	*-9*	*-8*	*-7*	*-6*	*-5*	*-4*	*-3*	*-2*	*-1*
3	8	8	6	6	6	-	-	-	-	-
4	9	8	6	7	6	-	-	-	-	-

From the above table it is deduced that if, for example, the owed quantity after the first visit of the vehicle at the site of the third customer is 8 items of the product then it is optimal to go to the depot to restock with 10 items, to return to the third customer to deliver 6 items and proceed to fourth customer. In this case the penalty cost is equal to $2*\pi_3 = 3.4$.

(b) Suppose that $N=8$ and $Q=8$. The travel costs between customers j and $j+1$, $j=1,\ldots,7$, are given by: $c_{12}=2$, $c_{23}=3$, $c_{34}=2$, $c_{45}=1$, $c_{56}=3$, $c_{67}=4$ and $c_{78}=2$. The travel costs between customers j, $j=1,\ldots,8$, and the depot are given by: $c_{10}=3$, $c_{20}=5$, $c_{30}=4$, $c_{40}=7$ $c_{50}=5$, $c_{60}=3$, $c_{70}=5$ and $c_{80}=3$. Note that these costs satisfy the triangle inequality. We assume that, for each customer $j=2,\ldots,8$, the penalty cost π_j is equal to 1.7. We further assume that for each customer $j=1,\ldots,8$, the demand ξ_j for the product follows the Binomial distribution $Bin(Q, p)$, i.e.

$$Pr(\xi_j = x) = \binom{Q}{x} p^x (1-p)^{Q-x}, x = 0,\dots,Q.$$

For $p = 0.3,$ in Table 4 below, we present the actions selected by the optimal policy for each customer $j,\ j = 1,\dots,7.$ Each row of the table corresponds to each customer $j,\ j = 1,\dots,7,$ and each column of the table corresponds to the number $z \in \{-8,\dots,0,\dots,8\}$ of items of the product.

Table 4. The actions selected by the optimal policy

j	*-8*	*-7*	*-6*	*-5*	*-4*	*-3*	*-2*	*-1*	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
1	-	-	-	-	-	-	-	-	2	2	1	1	1	1	1	1	1
2	4	4	3	3	2	2	2	2	2	2	1	1	1	1	1	1	1
3	4	4	3	3	3	3	3	2	2	2	1	1	1	1	1	1	1
4	3	3	3	3	3	3	3	1	1	1	1	1	1	1	1	1	1
5	3	3	3	3	3	1	1	1	1	1	1	1	1	1	1	1	1
6	3	3	3	3	3	3	3	1	1	1	1	1	1	1	1	1	1
7	3	3	3	3	1	1	1	1	1	1	1	1	1	1	1	1	1

In Table 5 below, we present the optimal value of $\theta,$ for customer $j \in \{2,\dots 7\}$ when the optimal action is action 3. The symbol "-" indicates that action 3 is not optimal.

Table 5. The value of optimal θ for customer $j \in \{2,\dots 7\}$

j\z	*-8*	*-7*	*-6*	*-5*	*-4*	*-3*	*-2*	*-1*
2	-	-	5	5	-	-	-	-
3	-	-	4	4	3	3	2	-
4	6	6	6	3	4	3	2	-
5	5	5	5	5	4	-	-	-
6	4	5	6	4	4	2	2	-
7	6	6	5	5	-	-	-	-

The value of the minimum total expected cost during a visit cycle is equal to $f_0 = c_{0,1} + Ef_1(Q - \xi_1) = 24.789.$

Concluding remarks

In this paper we proposed a dynamic programming method for a particular capacitated vehicle routing problem in which a single vehicle starts its route from a depot and delivers a product to *N* customers according to a particular order. The demands of the customers for the products are stochastic and each customer's demand is less than or equal to the vehicle capacity. A customer's demand may not be satisfied or may be satisfied partially. We selected as decision epochs the epochs at which the vehicle visits for the first time each customer and has satisfied as much of the customer's demand as possible. The cost structure of the problem includes travel costs between consecutive customers, travel costs between customers and the depot and penalty costs due to unsatisfied demands. It is proved that the policy that minimizes the expected total cost divides the set of all possible loads carried by the vehicle after the first visit to each customer into five disjoint subsets. If the load of the vehicle belongs to the first set, then the optimal decision is to proceed to the next customer. If it belongs to the second subset, then it is optimal to go to the depot for restocking, and then to proceed to the next customer. If it belongs to the third subset, then it is optimal not to satisfy the remaining demand and to proceed directly to next customer or to go to the depot for restocking and then proceed to next customer. If it belongs to the fourth subset, then it is optimal to go to the depot for restocking, to return to the customer in order to satisfy fully or partially the remaining demand and then proceed to next customer. If it belongs to fifth subset, then it is optimal to go to the depot to restock the owed quantity, to return to the customer to deliver the owed quantity, to make a second trip to the depot for restocking, and then to go to next customer.

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