Lecture Notes in Management Science (2014) Vol. **6**: 150–158
6th International Conference on Applied Operational Research, Proceedings

ISSN 2008-0050 (Print), ISSN 1927-0097 (Online)

A transportation branch and bound algorithm for solving the generalized assignment problem

Elias Munapo

Graduate School of Business and Leadership, University of KwaZulu-Natal,
Westville Campus, Durban, South Africa
munapoe@ukzn.ac.za

Abstract. This paper presents a transportation branch and bound algorithm for solving the generalized assignment problem. This is a branch and bound technique in which the sub-problems are solved by the available efficient transportation techniques rather than the usual simplex based approaches. A technique for selecting branching variables that minimizes the number of sub-problems is also presented.

Keywords: generalized assignment problem, branch-and-bound method, transportation model

Introduction

The generalized assignment problem (GAP) is the problem of assigning *n* jobs to *m* tasks such that total cost is minimal and that each job is assigned to exactly one agent and subject to agent's capacity. GAP is NP hard and has had so many approaches being proposed in the past 50 years. This model is the general case of the assignment problem in which both tasks and agents have a size and the size of each task may change from one value to another. GAP has many applications in real life and these include vehicle routing: Toth and Vigo (2001), resource allocation: Winston and Venkataramanan (2003), supply chain, machine scheduling and location among others. It is because of these important applications that so many exact and inexact methods have been proposed. The exact approaches that were developed include methods by: Ross and Soland (1975); Martello and Toth (1981); Fisher *et al*. (1986); Guignard and Rosenwein (1989); Karabakal *et al*. (1992); Savelsburgh (1997) and Pigatti *et al*. (2005). An inexact method or heuristic is a method that gives a highly accurate but not necessarily optimal solution. Some of the heuristics for the generalized assignment problem were developed by: Laguna *et al*. (1995); Osman (1995); Chu and Beasley (1997); Asahiro *et al*. (2003); Nauss (2003) and Yagiura *et al*. (1998, 2004, 2006). Note that: Nauss (2003) presented both a heuristic and an optimizing approach. In this paper we propose a transportation branch and bound algorithm for solving the generalized assignment problem. This is a branch and bound technique in which the sub-problems are solved by use of the available efficient transportation techniques rather than the usual simplex based approaches. A technique for selecting branching variables so as to minimize sub-problems is also presented.

Generalized Assignment Problem

A mathematical formulation of the generalized assignment problem may be represented as shown in (1).

$$Z_{GAP} = \text{Minimize} \sum_{i}^{m}\sum_{j}^{n} c_{ij}x_{ij} \qquad (1)$$

Subject to:

$$\sum_{j}^{n} r_{ij}x_{ij} \le b_i, \forall i$$

$$\sum_{i}^{m} x_{ij} = 1, \forall j$$

where, $x_{ij} = 0$ or 1, $i = 1,2,...m,$ is a set of agents, $j = 1,2,...n,$ is a set of tasks, c_{ij} is the cost of assigning agent *i* to task *j*, r_{ij} is the resource needed by agent *i* to do task *j*, b_i is the resource available to agent *i*.

Relaxing the generalized assignment problem

The generalized assignment problem can be relaxed to become an ordinary transportation problem. A transportation model is easy to handle and efficient solution methods such network approaches are available. This type of relaxation was proposed by Munapo *et al*. (2010).

Relaxing the model

Some of the restrictive GAP constraints are given in (2),

$$\sum_{j}^{n} r_{ij}x_{ij} \le b_i, \forall i \qquad (2)$$

The GAP model can be relaxed by replacing these constraints with other forms of inequalities given in (3).

$$\sum_{j}^{n} x_{ij} \leq \gamma_i, \forall i \tag{3}$$

Thus the model becomes a transportation one as presented in (4).

$$Z_{relaxed} = Minimize \sum_{i}^{m}\sum_{j}^{n} c_{ij} x_{ij} \tag{4}$$

Subject to:

$$\sum_{j}^{n} x_{ij} \leq \gamma_i, \forall i$$

$$\sum_{i}^{m} x_{ij} = 1, \forall j$$

Where γ_i is obtained by solving the knapsack problem in (5).

$$\gamma_i = \text{Maximize} \sum_{j}^{n} x_{ij} \tag{5}$$

Subject to:

$$\sum_{j}^{n} r_{ij} x_{ij} \leq b_i$$

$$x_{ij} = 0 \text{ or } 1$$

The optimal solution to this knapsack problem is readily available.

Solving the knapsack problem

The optimal solution to the knapsack solution can be obtained by arranging the resource coefficients in row *i* in ascending order. i.e.,

$$r'_{i1}, r'_{i2}, \ldots, r'_{in} \tag{6}$$

where,

$$r'_{i1} \leq r'_{i2} \leq \ldots \leq r'_{i\gamma_i} < \ldots < r'_{in} \tag{7}$$

are the arranged coefficients. The knapsack objective value γ_i, is the largest integral value such that

$$b_i \geq r'_{i1} + r'_{i2} + \ldots + r'_{i\gamma_i} \tag{8}$$

where, $1 \leq \gamma_i \leq n$. The integral value γ_i is now the supply in the transportation model.

The transportation model

The optimal solution to the transportation model will act as a lower bound to the generalized assignment problem and is usually infeasible to the original GAP model. The relaxed problem is shown in Table 1.

Table 1. Transportation problem

					Supply
	c_{11}	c_{12}	…	c_{1n}	γ_1
	c_{21}	c_{22}	…	c_{2n}	γ_2
	…	…	…	…	
	c_{m1}	c_{m2}	…	c_{mn}	γ_m
Demand	1	1		1	

This transportation problem is not a balanced model. In most cases

$$\sum_{i}^{m} \gamma_i \neq n \tag{9}$$

If

$$\sum_{i}^{m} \gamma_i < n \tag{10}$$

Then (1) is infeasible. i.e., at least one of constraints, $\sum_{i}^{m} x_{ij} = 1,$ is violated.

If

$$\sum_{i}^{m} \gamma_i = n \tag{11}$$

Then the relaxed model can be solved directly without balancing. The solution to the relaxation is optimal if it satisfies (1).

If

$$\sum_{i}^{m} \gamma_i > n \quad \text{or} \quad \sum_{i}^{m} \gamma_i < n \tag{12}$$

Then the relaxed model requires balancing before applying transportation techniques. To balance the transportation problem, a dummy column is added when we have $>$ while a dummy row is suitable when we have $<$. When the transportation is balanced then the optimal solution can be found by using network codes for transportation models. These are efficient and recommended and the sub problems are not solved from scratch. The current solutions are used as starting solutions in the next iterations. Lagrangian or linear programming (LP) relaxations are not readily useful to this procedure. With this approach it is only possible to branch if the relaxation gives an integer optimal solution and this is not possible with LP or Lagrangian relaxations.

Branch and Bound Approach

A branch and bound method can be used to ascend from the lower bound to an optimal solution of the generalized problem. The lower bound obtained by solving the relaxed model is usually infeasible to (1). A row *i* that is not feasible can be selected, a clique inequality generated and used to create branches.

Generating branching inequalities

Suppose from row *i,* the following variables are basic and they make up an infeasible solution

$$x_{if1}, x_{if2}, \ldots, x_{ifl} \tag{13}$$

$$r_{if1} + r_{if2} + \ldots + r_{if\ell} > b_i \tag{14}$$

where x_{ifj} is basic variable and r_{ifj} is its corresponding resource coefficient with $j = 1,2,\ldots\ell$.

From the inequality given in (14), it implies that some of these basic variables are not supposed to be basic. One or more of these basic variables may not be basic and the exact number is only known for specific problems. Branching does not necessarily mean the transportation sub-problem has to be resolved from scratch. The sub-problem is solved by improving the current solution. The previous solution is used as a starting solution in the next iteration.

Order of branching

The order of branching is very important as it can affect the size of the search tree. Strategies are required to determine a branching order that results in the smallest search tree. In this paper it is recommended that branching starts with those rows that have the least number of choices. In other words the most restricted rows are used in creating branches: Kumar *et al* (2007). Branching must always start with the most restricted row. The most restricted row in this paper is defined as that row where the least number of branches can be generated.

Transportation branch and bound algorithm for GAP

The transportation branch and bound algorithm for the generalized assignment problem consists of the following steps,

- **Step1:** Relax GAP to obtain a lower bound.

- **Step 2:** Select the most restricted row to come up with branching variables.
- **Step 3:** Branch using the selected variables. Return to step 2 until the best transportation solution is feasible.
- **Best solution:** A solution is said to be the best solution if it is the smallest optimal solution available.

Optimality

Suppose the terminal nodes are given in (20).

$$Z_1^T \; Z_2^T \ldots Z_\eta^T \tag{20}$$

The upper bound is selected from the node giving best solution so far.

$$Z_{GAP} = \min[Z_1^T, Z_2^T, \ldots, Z_\eta^T] \tag{21}$$

Thus Z_{GAP} is optimal.

A node is said to be a terminal one if the

- optimal solution to the transportation model is feasible to original GAP model,
- transportation model does not have feasible optimal solution or
- optimal solution to the transportation model is bigger than a given upper bound.

Note:

Generation of clique inequalities and using them as cuts is not a new idea. Clique constraints used in this paper are in fact a simple type of knapsack constraints generated from single constraints of the original problem. Knapsack constraint generators are very common in modern MIP solvers. What is new is the fashion of using these inequalities to form branches and solving the sub problems generated as transportation problems. This is effective for GAP models. *Jumptracking* is preferred in this procedure and branching is done on a node with the smallest objective value.

Numerical illustration

Use the transportation branch and bound algorithm to solve the following GAP model.

$$Z_{GAP} = Minimize \tag{22}$$

$$28x_{11} + 76x_{12} + 52x_{14} + 28x_{15} + 98x_{21} + 40x_{23} + 92x_{24} + 98x_{25} + 90x_{32} + 32x_{33} + 20x_{34}$$

Subject to: (23)

$$24x_{11} + 38x_{12} + 22x_{14} + 36x_{15} \le 56; 12x_{21} + 22x_{23} + 30x_{24} + 36x_{25} \le 56$$
$$20x_{32} + 28x_{33} + 44x_{34} \le 56; x_{11} + x_{21} = 1; x_{12} + x_{32} = 1; x_{23} + x_{33} = 1;$$
$$x_{14} + x_{24} + x_{34} = 1; x_{15} + x_{25} = 1; x = 0 \; or \; 1 \; \forall ij.$$

Table 2. Transportation model for numerical illustration

						Supply
	28	76	L	52	28	2
	98	L	40	92	98	2
	L	90	32	20	L	2
Demand	1	1	1	1	1	

The letter **L** shows that an assignment is not possible in that cell. A dummy column is introduced to balance the transportation problem as shown in Table 3.

Table 3. Balancing the transportation model (by adding a dummy column)

							Supply
	28	76	L	52	28	0	2
	98	L	40	92	98	0	2
	L	90	32	20	L	0	2
Demand	1	1	1	1	1	1	

Note:

- A terminal node is said to be feasible if the optimal solution to the transportation sub-problem is feasible to the original GAP problem.
- A terminal node is said to be infeasible if the optimal solution to the transportation is infeasible to the GAP model.
- DNE means the transportation sub-problem does not have a feasible optimal solution
- The numbers in the circles denote the order of solution.

From the search tree given in Figure 1 the optimal solution to the GAP problem is given as shown in (24)

$$Z_{GAP} = \min[Z_1^T, Z_2^T, Z_3^T, Z_4^T, Z_5^T, Z_6^T] = 300 \quad (24)$$

$$x_{11} = x_{14} = x_{25} = x_{32} = x_{33} = 1$$
$$x_{12} = x_{15} = x_{21} = x_{23} = x_{24} = x_{32} = x_{33} = x_{34} = 0 \quad (25)$$

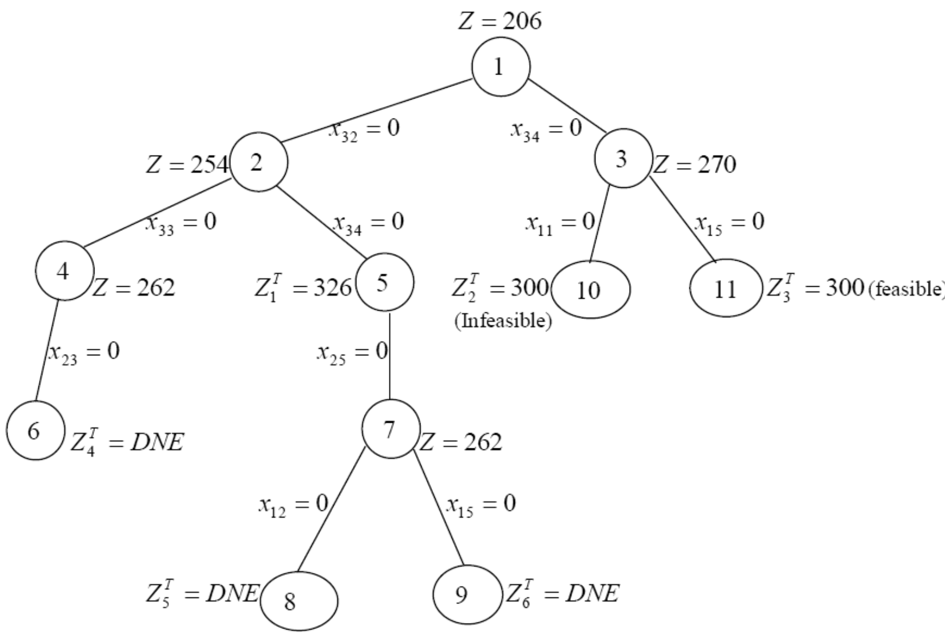

Fig. 1. Full search tree for the given numerical illustration

Conclusions

The proposed approach has the advantage that the individual, γ_i, values can be found independently allowing the much needed use of parallel processors. The sub-problems resulting from the search trees are transportation models and can be solved efficiently by the available network approaches. The sub-problems that result from the usual branch and bound related approaches are NP hard integer models which are very difficult to solve. The only nuisance to this approach is that like the simplex based approaches it is also not spared by degeneracy. In the search tree given in Figure 1, it can be noted that there is no change in the objective value from node 4 to node 7. The degeneracy drawback can be alleviated by noting all alternate optimal solutions at every node and then branch in such a way that the objective value does not remain static. Attempts will be made in future to use cuts in branching and compare its efficiency with the available approaches and explore for better strategies that can significantly improve the selection of branching variables.

References

Asahiro Y, Ishibashi M and Yamashita M (2003). Independent and cooperative parallel search methods for the generalized assignment problem. Optimization methods and Software 18(2): 129-141

Chu PC & Beasley JE (1997). A genetic algorithm for the generalized assignment problem. Computational Operations Research 24: 17-23

Fisher ML, Jaikumar R and Van Wassenhove LN (1986). A Multiplier Adjustment Method for the Generalized Assignment Problem. Management Science 32(9): 1095-1103

Guignard M and Rosenwein M (1989). An Improved Dual-Based Algorithm for the Generalized Assignment Problem. Operations Research, 37(4): 658-663

Karabakal N, Bean JC and Lohmann JR, (1992). A steepest Descent Multiplier Adjustment Method for the Generalized Assignment Problem. Report 92-11, University of Michigan, Ann Arbor, MI.

Kumar S, Munapo E and Jones BC (2007). An Integer Equation Controlled Descending path to a Protean Pure Integer Program. Indian Journal of Mathematics 49: 211-237

Laguna M, KellY JP, Conzalez-Velarde JL & Glover FF (1995). Tabu search for the generalized assignment problem. European Journal of Operational Research 82: 176-189

Martello S and Toth P, 1981, An algorithm for the generalized Assignment Problem, In Operations Research 81, Brans JP (ed.), North-Holland, Amsterdam: 589-603

Munapo E, Kumar S and Musekwa SD (2010). A revisit to the generalized assignment problem: an ascending hyper-plane approach through network flows. International Journal of Management Prudence 1(2): 7-14

NAUSS RM, 2003, Solving the generalized assignment problem: An optimizing heuristic approach. INFORMS Journal of Computing 15(3): 249-266

Osman IH (1995). Heuristics for the generalized assignment problem: Simulated annealing and tabu search approaches. OR Spektrum 17: 211-225

Pigatti A, Poggie De Aragao M and Uchoa E (2005). Stabilized branch and cut and price for the generalized assignment problem. In 2nd Brazilian Symposium on Graphs, Algorithms and Combinatorics. Electronic Notes in Discrete Mathematics, Elsevier, Amsterdam 19: 389-395

Ross GT and Soland RM (1975). A Branch and Bound algorithm for the Generalized Assignment Problem. Mathematical Programming 8: 91-103

Savelsburgh M (1997). A branch and price algorithm for the generalized assignment problem. Operations Research 45(6): 831-841

Toth P and Vigo D (2001). The Vehicle Routing Problem, Philadelphia: SIAM Monographs on Discrete Mathematics and Applications.

Winston WL and Venkataramanan M (2003). Introduction to Mathematical Programming Thomson Brooks/Cole, 4th edition.

Yagiura M, Ibaraki T and Glover F (2004). An ejection chain approach for the generalized assignment problem. Informs Journal of Computing 16: 133-151

Yagiura M, Ibaraki T and Glover F(2006). A path re-linking approach with ejection chains for the generalized assignment problem. European Journal of Operational Research 169: 548-569

YAGIURA M, Yamaguchi T and IBARAKI T (1998). A variable depth search algorithm with branching search for the generalized assignment problem. Optimization Methods & Software 10: 419-441.