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Optimal connections on a network with multiple interchangeable origins and multiple destinations

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Abstract

On the basis of counter-examples, it is demonstrated that the time-honoured successive shortest path strategy fails to achieve a minimal total length of the transportation routes on a network with multiple interchangeable origins and multiple destinations. In certain cases the successive shortest path strategy fails to achieve a minimal total length of the transportation routes even for multiple interchangeable origins and a single destination. This is due to the appearance of cyclic paths with dominating flow in one of the directions of traversal called dominated parasitic flow loops. Removing all dominated parasitic flow loops does guarantee a minimal total length of the transportation routes and is the key to developing efficient transportation and communication networks. The probability of existence of dominated parasitic flow loops in networks is surprisingly high and a state where such flow loops are present is the 'natural state' of real networks. Accordingly, a new algorithm for eliminating all dominated flow loops in networks has been proposed, based on transforming the initial network into a single source/single sink network connected with the rest of the network through edges with unit capacities, followed by maximising the flow in the transformed network at a minimum cost. In addition, a technique referred to as 'repeated inversion of the flow of dominated parasitic flow loops' is proposed and successfully applied to reduce the total length of transportation routes.

Introduction

The performance of many route planners depends exclusively on building successive shortest paths on the road map. A shortest-path from an origin to a destination is usually determined by one of the well-known classical algorithms: the *Dijkstra algorithm* or the *Bellman-Ford* algorithm or their modifications. To minimise the total cost of a maximum flow between a single origin and a single destination Goodrich and Tamassia (2002) proposed building repeated successive shortest paths. Repeated building of shortest paths has, for example, been adopted in (Wang *et al*, 2005) to minimise the total length of paths between multiple origins and multiple destinations in solving the multiple pairs shortest-path problem. Repeated building shortest path from multiple origins to a common destination to minimise the total length of routes has recently been proposed in Zhao and Wang (2014). Unfortunately, the successive shortest path strategy for minimising the total length of routes from origins to destinations results in *dominated parasitic flow loops* in the solutions which render them far from optimal. *A dominated parasitic flow loop is a cyclic path, more than half of whose length is occupied with flow in one of the directions of traversal*.

Building optimal routes from multiple interchangeable origins to multiple destinations has numerous important practical applications. It covers transportation and delivery services from interchangeable supply centres, gas supply to consumers from different storage locations, fuel products supply from different terminals to petrol stations, vehicle routing with real-time traffic information (taxi car fleets servicing customers placing calls from different locations; ambulance cars from different hospitals servicing calls from different locations etc. Common services provided by professionals living at different locations are also covered by this case. A related problem is the problem of optimal matching of sources and destinations on a weighted bipartite graph which minimises the total transportation cost. Various algorithms for solving this problem have recently been reviewed in (Sharathkumar and Agarval, 2012). The delivery cost from a source to a destination however, is fixed and cannot be altered in the process of finding an optimal solution.

Parasitic flow loops in networks are associated with increased risk of congestion and accidents, wastage of energy, time, and increased levels of pollution to the environment. It has already been shown that closed parasitic flow loops exist in real networks with a very high probability (Todinov 2013a). The existence of closed parasitic flow loops in networks whose throughput flow has been maximised, remained unnoticed for nearly 60 years. Ironically, despite the decades of intensive research on maximising the flow in networks (Asano & Asano, 2000), closed parasitic flow loops appear in the "network flow solutions" of almost all published algorithms since the creation of the theory of flow networks in 1956. These algorithms include the classical Ford-Fulkerson algorithm (Ford and Fulkerson, 1956), the Edmonds and Karp algorithm (Edmonds and Karp, 1972), the Goldberg's algorithm (Goldberg and Tarjan 1988) and in more recently developed algorithms such as the draining algorithm in (Dong *et al*, 2009).

Indeed, consider the counter-example network in Fig.1 (Todinov 2013c), where all edges have capacity equal to 10 flow units per unit time. The classical Edmonds and Karp shortest-path algorithm proceeds with saturating the shortest-path (1,2,3,13) with 10 units of flow, followed by saturating the next shortest path (1,5,6,3,4,13) with 10 units of flow and finally, with saturating the path (1,8,9,10,11,4,2,7,12,13) with 10 units of flow. As a result, a closed flow loop (2,3,4,2) appears, carrying 10 units of flow which effectively never leaves the network. The closed parasitic flow loop can be removed safely, without affecting the total throughput flow of 30 units from the source *s* to the sink *t*.

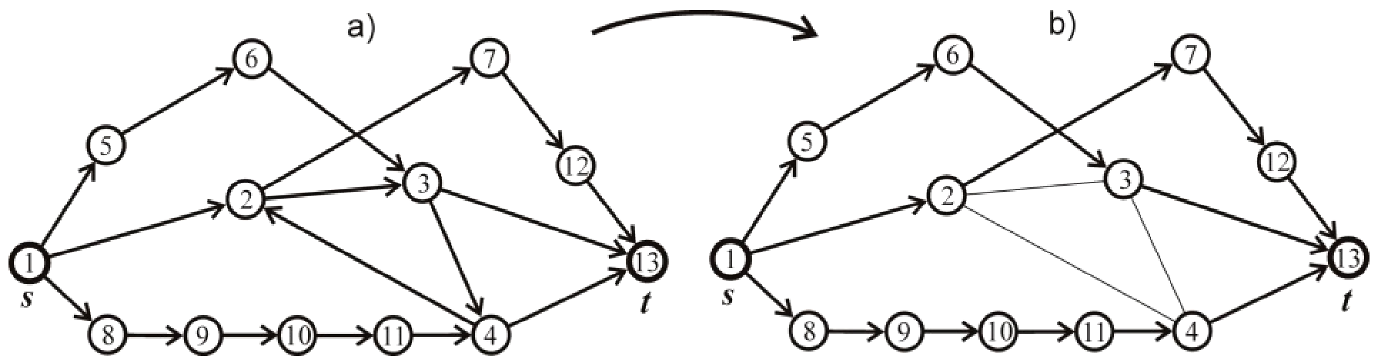


Figure 1. A counterexample network, demonstrating that the classical Edmonds and Karp (1972) shortest-path algorithm leaves a closed parasitic loop of flow (4,2,3,4) in the optimised network. All edges have a flow capacity of 10 units (Todinov 2013c).

To the best of our knowledge, no correct strategy has yet been proposed for minimising the total length of connections in the very important cases of multiple interchangeable sources and multiple destinations and multiple interchangeable sources and a single destination. This significant gap defines the objectives and the main contributions of this paper:

Eliminating dominated flow loops by a repeated inversion of dominant flows preserving an invariant

Cyclic paths in which more than half of the path length is occupied by flow along a particular direction of traversal are associated with significant transportation losses and risk of congestion. Such flow loops will be referred to as '*dominated flow loops*'. The closed parasitic loop of flow is always a dominated flow loop but the converse is not true in general. Repeated inversion of the direction of flows along cyclic paths with dominating flow can be used to reduce the total length of transportation routes/connections between multiple interchangeable origins and multiple destinations. Optimizing supply networks by draining highly undesirable dominated parasitic flow loops derives significant value by reducing the transportation costs, the risk of congestion and accidents, and the environmental pollution. The end result is a huge amount of resources saved to the world economy.

Consider Figure 2, where three interchangeable origins s1,s2 and s3 are supplying a particular commodity or service to three destinations d1,d2 and d3. A unit flow from each origin to each destination represents the existing connections. As a result, a dominated parasitic flow loop 4,5,6,7,2,3,4 appears. The dominated parasitic flow loop can be eliminated by pushing through the cyclic path 4,5,6,7,2,3,4 a unit flow in the opposite direction to the direction of the dominating flow (in a direction 2,7,6,5,4,3,2). As a result, the dominated parasitic flow loop disappears (Figure 2b) without interrupting connections between the interchangeable origins and the destinations. It can be shown that for any set of interchangeable origins connected to a set of destinations, pushing uniform flow in the opposite direction of the flow of the dominated flow loop always maintains the invariant*: each interchangeable origin services exactly one destination and each destination is being serviced by exactly one origin*. As a result, this technique will be referred to as *repeated inversion of the dominating flows preserving an invariant*.

In addition, it can be shown that for a set of interchangeable origins and a set of destinations, the necessary and sufficient condition for the shortest total length of the connections between the origins and the destinations is *the non-existence of dominated parasitic flow loops*. The proof is omitted due to a lack of space. Repeated inversion of the dominating flows along dominated parasitic flow loops is then conducted until no more dominated parasitic flow loops can be found. Because during this operation the invariant property ("each interchangeable origin is connected to exactly one destination") is preserved, the obtained solution is a feasible solution and an optimal solution.

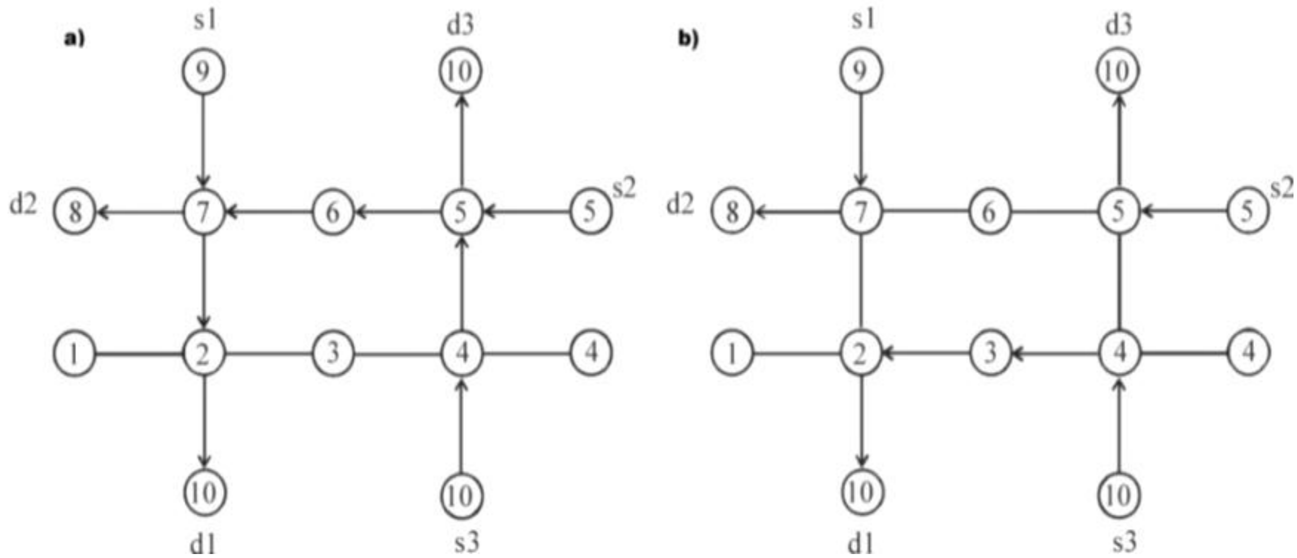


Figure 2. Draining a dominated flow loop (2,7,6,5,4,3,2)

Consider the network in Fig.3a which features three interchangeable origins s1, s2 and s3, connected to three destinations d1,d2 and d3. Suppose that the lengths of the edges represent the distances between the nodes. The shortest path from s1 to d1 is the edge (1,2); the shortest path from origin s2 to destination d2 is the edge (3,4) and the shortest path from the remaining origin s3 to the remaining destination d3 is (5,6,7).

The cyclic path (1,2,4,6,7,1 however, is a cyclic path with dominating flow and can be drained by augmenting it with a unit flow in the opposite direction. As a result, the dominating flow disappears from the cyclic path while the connections between origins and destinations are preserved (Fig.3b). In Fig.3b, the cyclic path (5,6,4,5) is with dominating flow which can be drained by augmenting the cyclic path with unit flow in direction (5,4,6,5). This operation results in the network from Fig.3c where no cyclic paths with dominating flow are present and the total length of connections for delivering the service from the interchangeable origins to the destinations is the smallest possible. The congestion associated with this transportation network is also minimised. In the example from Fig.3, the successive shortest-path strategy failed to find the optimal solution.

Selecting the nearest available origin, for each destination, also does not guarantee an optimal solution. In the network from Figure 4a, including the interchangeable origins s1,s2,s3 servicing destinations d1,d2,d3, the nearest origin to destination d1 is s1; the nearest origin to destination d2 is s2 and the nearest remaining origin to destination d3 is s3. In addition, all of the origin-destination pairs have been connected with the shortest paths. Despite the shortest-path selections, the obtained solution is far from optimal. *A* dominated parasitic flow loop 12,8,3,6,5,11,12 is present and by augmenting it with unit flow in the opposite direction of the direction of the dominant flow, the new, optimal set of connections in Figure 4b appear where no dominated parasitic flow loops are present. The connections of the interchangeable origins s1,s2 and s3 with the destinations d1,d2 and d3 have been guaranteed while the total length of transportation routes has been reduced significantly.

Suppose now that nodes 11,8 and 6 in Fig.4a are connected through edges with the same length and a limited throughput capacity of 10 units to a common destination *d* which demands 30 units of flow per unit time. As can be verified, the successive shortest path strategy in connecting the sources s1,s2 and s3 with the common destination *d* leads to the appearance of the same dominated parasitic flow loop 12,8,3,6,5,11,12. As a result, in networks with multiple origins and a single destination, characterised by unlimited throughput capacity of the edges, the successive shortest path strategy fails to minimise the total length of the routes if the throughput capacity of the immediate delivery edges is limited.

The process of augmenting dominated parasitic flow loops, until no dominated parasitic flow loops can be found, will end after a finite number of steps. Indeed, the edges carrying network connections can be thought as edges saturated with a certain quantity of flow units. Because the lengths of the edges can always be expressed as rational numbers, multiplying the lengths of the edges by their common denominator will express them with integer numbers.

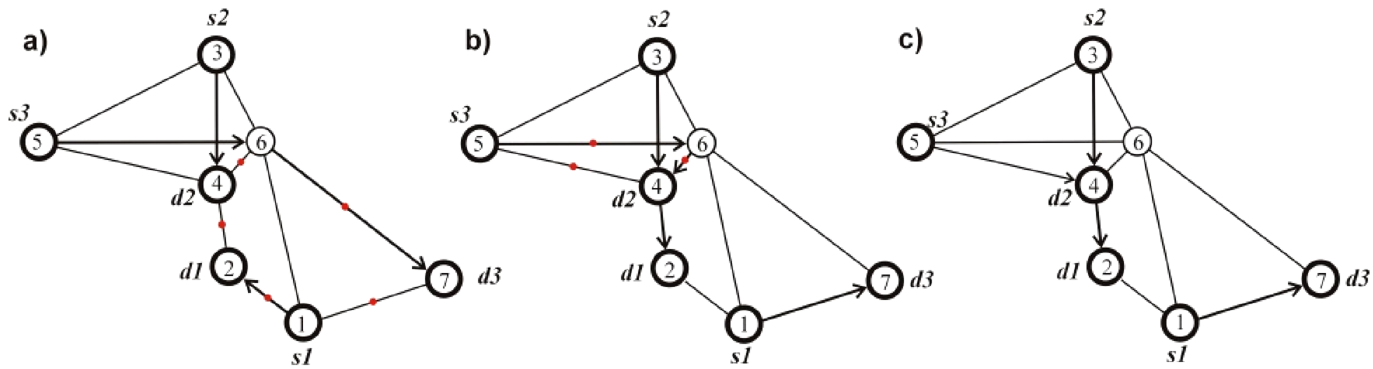


Figure 3. Repeated draining of dominated flow loops by reversing the dominating flow ensures a minimal total length of the connections and is a way of drawing value from existing supply networks.

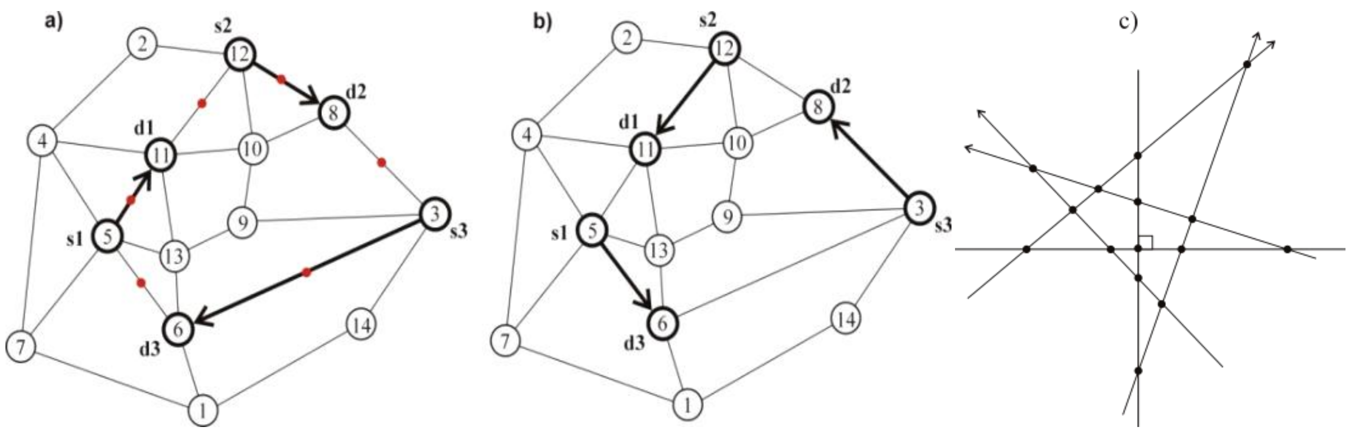


Figure 4. a) Contrary to the conventional wisdom, selecting the nearest available origin to each destination does not guarantee an optimal solution. b) The optimal solution is obtained only after eliminating the cyclic paths with dominating flow. c) A network formed by the intersection nodes of randomly oriented flows.

Suppose that there are *m* edges in the network and the maximum possible length of an edge is $u_{\max}$ units. The total length of connections between origins and destinations therefore, cannot exceed $mu_{\max}$ length units. Because during the flow inversion, reversed flow is pushed only through cyclic paths with dominating flow, and after each flow inversion a dominated parasitic flow loop ceases to exist, each augmentation of a parasitic flow loop reduces the total length of connections by at least one length unit. Because there can be at most $mu_{\max}$ length units, after a finite number of steps, there will be a step at which no dominated parasitic flow loops will be present.

The probability of existence of dominated flow loops in networks is surprisingly high. Consider the system of intersecting paths on a plane depicted in Fig.4c: two perpendicular linear paths which do not carry any flow and *n* randomly oriented linear flow paths each of which carries flow to its full capacity.

It can be shown (the proof is omitted due to a lack of space) that for this system, the probability of existence of a dominated flow loop between the nodes formed by the intersecting lines is $p = 1 - 1/4^{n-1}$ — a value which is very close to unity even for small values of *n*. As a result, the state where dominated flow loops are present can be considered to be the 'natural state' of real flow networks. This emphasizes the importance of an algorithm for removing dominated parasitic flow loops from networks. Such an algorithm is presented next.

At the first step of the algorithm, the flow network with multiple origins $s_1, s_2, ..., s_n$ and destinations $d_1, d_2, ..., d_n$ (Fig.5a) is reduced to a flow network with a single super-source *s* (Fig.5b) and a single super-destination *t*. The multiple origins in Fig.5a are replaced by a single super-source with infinite flow generation, feeding each of the initial origins through lines whose capacity is exactly one flow unit. In a similar fashion, the multiple destinations in Fig.5a are replaced by a single super-destination connected with each of the separate destinations through edges whose capacity is exactly one flow unit. The sources of flow (the origins) and the destinations become ordinary throughput edges, with flow capacities equal to one. As a result, the origins of interchangeable commodity and the destinations 'disappear', and instead, ordinary throughput edges appear.

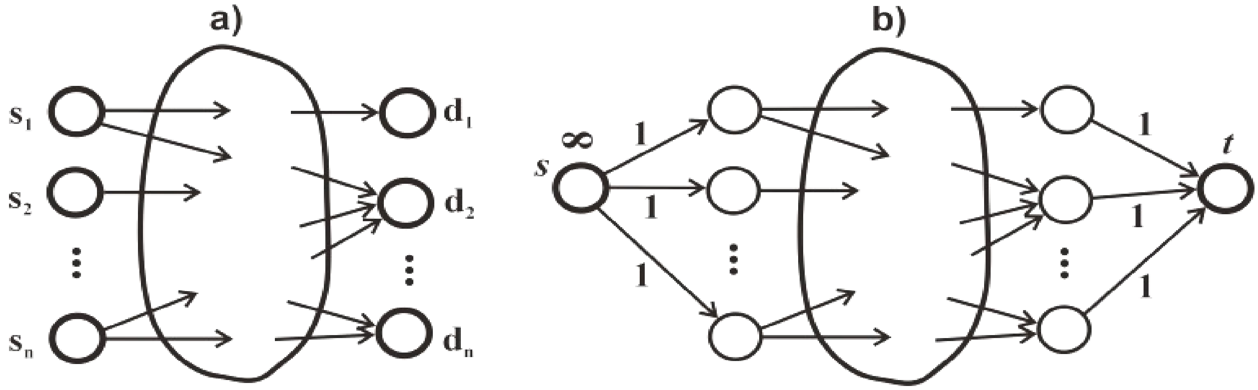


Figure 5. a) An example of a flow network with n interchangeable sources and n destinations. b) The network (a) has been reduced to a flow network with a single super-source s, and a single super-destination t.

The second step consists of assigning zero length to the edges connecting the super-source s and the super-destination t with the rest of the network. For the rest of the edges, the length of the edges corresponds to the real distances between the nodes.

The third step consists of maximising the flow in the transformed network (Fig.5b), from the super-source s to the super-destination t, at a minimum total transportation cost. The transportation cost associated with the separate edges is assumed to be proportional to their length. A number of algorithms have already been proposed for maximising the flow at a minimum total cost, some of which are characterised by a strictly polynomial running time (Tardos 1985; Orlin 1993). The method based on modified weights has been used for maximising the throughput flow at a minimum cost (see details in Todinov 2013b). It has worst-case running time $O(|f^*|m\log n)$ where $|f^*|$ is the magnitude of the throughput flow, m is the number of edges and n is the number of nodes in the network. Because the interchangeable origins are connected with the super-source s through edges with unit capacity, and the number of these edges does not exceed n, the worst-case running time of the algorithm for removing all dominated parasitic flow loops is $O(nm\log n)$.

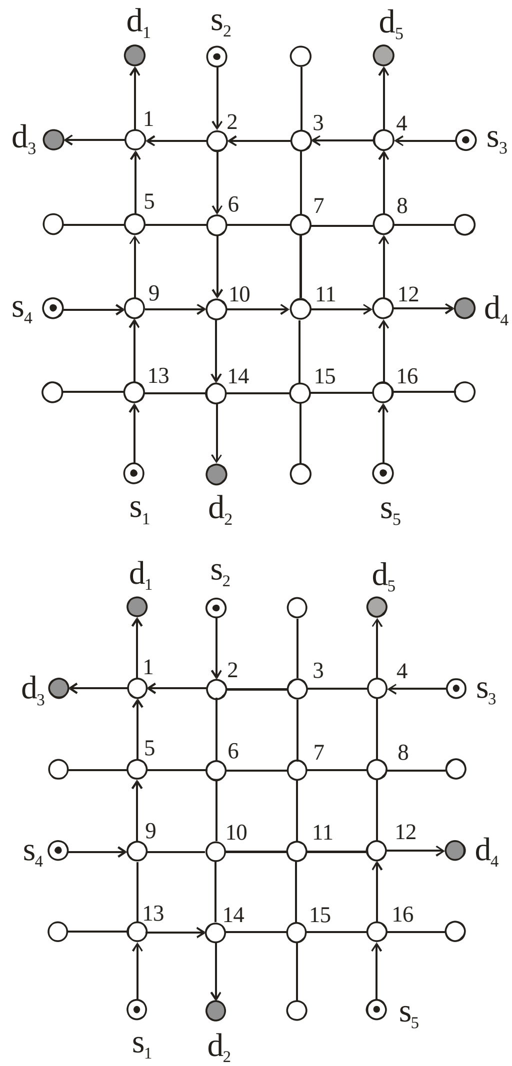
At the last step, the super-source s, the super-destination t and their connecting edges are removed from the transformed network. The nodes connected to the super-source and the super-destination become origins and destinations again. The edge flows in the resultant network define the optimal solution.

Correctness of the proposed algorithm

After conducting the third step of the algorithm, it can be shown that there are no dominated parasitic flow loops left in the network. Indeed, suppose that the specified maximum throughput flow of n units has been guaranteed and the total sum of the length of connections between origins and destinations is the smallest possible. Suppose that there exists a dominated parasitic flow loop in the network. In this case, the dominated flow loop can be augmented in a direction opposite to the dominating flow. After the operation, each of the origins will be connected to a destination but the total length of the connections will be reduced, therefore the total transportation cost will be reduced. However, this contradicts the fact that after conducting the third step of the algorithm, the total transportation cost associated with the obtained edge flows is the smallest possible. Consequently, there can be no dominated parasitic flow loops in the network after the third step.

Here is an illustrating example. Consider 5 volunteers belonging to the same organisation and living in different parts of a city allocated to 5 patients also living in different parts of the city (Fig.6a). The services of the volunteers are interchangeable and each volunteer must be assigned to exactly one patient. For the sake of simplicity, it has been assumed that all edges have the same length. One way of satisfying the demand from patients is shown in Fig.6a. Each origin has been connected with the corresponding destination through a direct, straight-line path. If the cost of transportation per edge is 100 units, the total cost of transportation to all destinations is 2500 units.

The solution produced by the proposed algorithm is shown in Fig.6b. The resultant network is characterised by a total transportation cost of 1500 units only, which constitutes a 40% reduction of the initial total transportation cost. As can be verified, no dominated parasitic flow loops exist, hence, the obtained transportation plan is associated with the smallest total transportation cost.



(a)

(b)

Figure 6. A real flow network a) before the optimisation and b) after the optimisation.

Conclusions

On the basis of counter-examples, it has been demonstrated that the traditional successive shortest path strategy does not guarantee a minimal total length of the transportation routes between multiple interchangeable origins and multiple destinations. In certain cases, this strategy does not guarantee a minimal total length of the transportation routes even for multiple interchangeable origins and a single destination. Removing all dominated parasitic flow loops does guarantee a minimal total length of the transportation routes and is the key to developing efficient transportation and communication networks.

A new algorithm for minimising the total length of the transportation routes by eliminating all dominated parasitic flow loops has been proposed. A technique referred to as 'repeated flow inverting preserving an invariant' has been proposed and successfully applied to minimise the total length of transportation routes between multiple interchangeable origins and multiple destinations. The probability of existence of dominated parasitic flow loops in networks is surprisingly high and the state where dominated parasitic flow loops are present can be considered to be the 'natural state' of flow networks.

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